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The Painlevé classification, dominant truncations and resonance analysis

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Abstract. A classification of ODE from the point of view of singular point analysis is suggested. The general formula for the Painlevé resonances is derived and the resonance analysis of low-order dominant truncations is performed. The resonance formulae for the Painlevé chains are presented and the significance of the Painlevé chains for the classification is explained.

1. Introduction

Painlevé-type equations, i.e. those whose solutions have no movable critical points [1], have become rather popular recently due to their connection with PDE integrable by the inverse scattering transform [2]. The classification of Painlevé-type ODE of the first, second and third order is known [1, 3] but to my knowledge there is no systematics of higher-order equations.

A way towards such systematics was indicated in [4, 5] where several special cases were investigated. The idea consists in using the singular point analysis suggested in [2] as a tool for the identification of candidates for higher-order Painlevé-type equations.

Singular point analysis (SPA) is a very simple and powerful method for testing the Painlevé property of differential equations. Basically it consists in substituting the anticipated expansion of a solution in the vicinity of a singular point x_0

$$u(x) = \sum_{n=0}^{\infty} u_n (x - x_0)^{n+p} \qquad p < 0 \tag{1.1}$$

into the tested equation and investigating whether this expansion is compatible with the equation and contains a sufficient number of undetermined coefficients for the approximation of a general solution. The equations that pass this test will be called Painlevé admissible.

In this paper a method for the classification of the Painlevé admissible ODE is presented and analysis of the 'resonances' of the expansion (1.1) is performed in detail.

The suggested classification follows the basic steps of the SPA. In the first step the general forms of dominant parts are specified. Next the resonance analysis is used to restrict coefficients of the dominant parts and finally the compatibility conditions of the SPA specify the 'recessive' parts of the equation.

The main topic of this paper is resonance analysis. We shall derive a resonance formula that will prove useful for many purposes, explaining, for example, the results on Painlevé chains in [4, 6].

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2. The dominant truncations

First, let us introduce some notation that will be used below. The dependent variable will be usually denoted by u and the independent one by x. The derivatives will be denoted as follows:

$$u_n = u_{nx} = u_{xx\ldots x} := \frac{\partial^n}{\partial x^n} u(x) \qquad n = 1, 2, \ldots \tag{2.1}$$

$$u_0 := u = u(x) \in \mathbb{C} \qquad x \in \mathbb{C}. \tag{2.2}$$

The polynomial ODE of order N can in general be written as

$$E(\varepsilon, g) := \sum_{K \in \varepsilon} g_K(x)[u]^K(x) = 0 \tag{2.3}$$

where ε is a set of multi-indices of length $N+1$,

$$K := (k_0, k_1, \ldots, k_N) \qquad k_i \in \mathbb{N} \tag{2.4}$$

$$[u]^K := u_0^{k_0} u_1^{k_1} \ldots u_N^{k_N} \tag{2.5}$$

and g_K are analytical functions.

The sum in (2.3) is supposed to be finite. We might admit $k_i \in \mathbb{Z}$ in (2.4) as well, which would correspond to rational ODE, but as any rational equation can always be multiplied by a common denominator there is no loss of generality in assuming $k_i \in \mathbb{N}$.

Example. The set of indices of the Riccati equation is

$$\varepsilon = \{(0, 1), (2, 0), (1, 0), (0, 0)\} \tag{2.6}$$

The first two steps of the SPA, i.e. the determination of the dominant behaviour and resonances, deal only with the so-called leading or dominant terms. These are the terms of the investigated equation that, after the substitution (1.1), produce the lowest power of $x - x_0$. It is therefore useful to define the dominant truncation of the ODE (we preserve the terminology of [4]).

Definition. The p dominance of the term $[u]^K$ is

$$D(p, K) := \sum_{j=0}^{N(K)} (p-j)k_j \tag{2.7}$$

where $N(K)$ is the order of the term $[u]^K$.

Definition. Let the ODE be of the form (2.3). The p dominance of the equation is

$$\mu(p, \varepsilon) := \min_{K \in \varepsilon} D(p, K). \tag{2.8}$$

The p-dominant truncation of the equation is

$$T(p, \varepsilon, g) := \sum_{\substack{K \in \varepsilon \\ D(p, K) = \mu(p, \varepsilon)}} g_K(x)[u]^K(x). \tag{2.9}$$

The most important are the dominant truncations for $p < 0$ which will be treated below.

The first step in the suggested classification of the Painlevé admissible ODE is to write down the classes of the dominant truncations with p dominance equal to m.

Definition. Let $M(p, m)$ be the set of all multi-indices K corresponding to terms (of *a priori* unspecified order) with the p dominance equal to m:

$$M(p, m) := \{K, D(p, K) = m\}. \tag{2.10}$$

Then

$$T(p, m, g) := \sum_{K \in M(p, m)} g_K[u]^K \tag{2.11}$$

where $g_K(x)$ are analytic functions. We shall denote the set of $T(p, m, g)$ with arbitrary (analytic) g by $T(p, m)$.

The simplest classes of p-dominant truncations $T(p, m)$ containing at least two terms are displayed in table 1.

Further classification of Painlevé admissible $T(p, m, g)$ follows from their resonance analysis.

Table 1. Dominant truncations with p dominance equal to m. The coefficients $A, B, C, \ldots$ are functions of x.

p	m	Element of $T(p, m)$
-1	-2	$Au_x + Bu^2$
	-3	$Au_{xx} + Bu_xu + Cu^3$
	-4	$Au_{xxx} + Bu_{xx}u + Cu_x^2 + Du_xu^2 + Eu^4$
	-5	$Au_{4x} + Bu_{xxx}u + Cu_{xx}u_x + Du_{xx}u^2$ $+ Eu_x^2u + Fu_xu^3 + Gu^5$
-2	-4	$Au_{xx} + Bu^2$
	-5	$Au_{xxx} + Bu_xu$
	-6	$Au_{4x} + Bu_{xx}u + Cu_x^2 + Du^3$
	-7	$Au_{5x} + Bu_{xxx}u + Cu_{xx}u_x + Du^2u_x$
-3	-6	$Au_{xxx} + Bu^2$
	-7	$Au_{4x} + Bu_xu$
-4	-8	$Au_{4x} + Bu^2$

3. The resonance formulae

The resonances corresponding to a differential equation are defined in [2] as powers of $(x - x_0)$ in the expansion (1.1) whose coefficients remain undetermined after the substitution of (1.1) into the equation. They are determined only by the dominant truncation of the equation.

Prior to the determination of the resonances of a given $T(p, m, g)$ we must evaluate the coefficient u_0 of the leading term in (1.1) The equation for $u_0 = a$ occurring in the leading-order term (1.1) of the solution expansion is

$$A(p, m, g, a) := \sum_{K \in M(p, m)} g_K(x_0)[p]^K a^{d(K)} = 0 \tag{3.1}$$

where $d(K)$ is the degree of the term $[u]^K$

$$d(K) := \sum_{j=0}^{N(K)} k_j \tag{3.2}$$

and

$$[p]^K := [p]_0^{k_0}[p]_1^{k_1}\dots[p]_N^{k_N} \tag{3.3a}$$

$$[p]_0 := 1 \qquad [p]_j := p(p-1)\dots(p-j+1) \qquad j\in\mathbb{N}_+. \tag{3.3b}$$

Equation (3.1) is obtained by the substitution $u(x)\approx a(x-x_0)^p$ in (2.11). Similarly, the equation for the resonances is obtained by the substitution

$$u = a(x-x_0)^p[1+b(x-x_0)^r] \tag{3.4}$$

into (2.11) and collecting terms linear in b [2]. In this way we obtain

$$T(p, m, g)\approx(x-x_0)^m A(p, m, g, a)+b(x-x_0)^{m+r}R(p, m, g, a, r). \tag{3.5}$$

The results can be summarised as follows.

Theorem 1. The resonances of the dominant truncation (2.11) are solutions of the equation for r:

$$R(p, m, g, a, r) := \sum_{K\in M(p,m)} g_K(x_0)[p]^K a^{d(K)}Y(p, K, r)=0 \tag{3.6}$$

where

$$Y(p, K, r) := \sum_{j=0}^{N(K)} k_j[r+p]_j/[p]_j \tag{3.7}$$

and a is a solution of (3.1).

As we can see, equation (3.6) is polynomial. Its degree is equal to the order of the dominant truncation $T(p, m, g)$. It depends on $u_0=a$ that on the other hand is a solution of (3.1). In general there can be more than one solution of (3.1) so we can get several families of resonances.

Corollary. In every family one of the solutions of (3.6) is always $r=-1$ (for any p, m, g).

Indeed

$$Y(p, K, r=-1)=p^{-1}D(p, K) \tag{3.8}$$

and due, to the fact that the sum in (3.6) is performed over K such that $D(p, K)=m$,

$$R(p, m, g, a, r=-1)=A(p, m, g, a) \tag{3.9}$$

and (3.6) is satisfied by $r=-1$ as a consequence of (3.1).

Actually, this corollary is nothing but a check of the correctness of (3.6), because it is a well known fact that the resonance $r=-1$ must appear as a consequence of arbitrariness of x_0 in (1.1).

In the following we shall use the resonance formulae (3.1) and (3.6) for the investigation of some simple dominant truncations.

4. The resonance analysis of $T(p, m, g)$

As explained in [2] an equation of order N can have the Painlevé property only if all resonances with $\mathrm{Re}\, r > 0$ are integer and for at least one solution a of (3.1) there are $N-1$ positive distinct integer roots of (3.6). This 'resonance value criterion' puts a restriction on possible $g_K(x)$ in the dominant truncations $T(p, m, g)$ obtained in § 2. The equations that have only one-term dominant truncations like, for example, $u_x = u$ must be treated separately and we shall not consider them here.

The simplest two-term dominant truncations belong to $T(-1, -2)$. They are of first order so they have only one resonance that inevitably must be -1 and no other conditions follow from the resonance criterion. This is in accordance with the fact that the general Riccati equation has the Painlevé property.

A more interesting case is $T(-1, -3)$ containing the dominant truncations

$$Au_{xx} + Buu_x + Cu^3 \tag{4.1}$$

where A, B, C are up to now arbitrary functions of x. The case $A = 0$ corresponds to $uT(-1, -2)$ where, moreover, the singular point $u = 0$ of the equation must be investigated. Therefore, we shall assume non-zero A and, without loss of generality, we can set $A = 1$. Equations (3.1) and (3.6) give for this case

$$C_0 a^2 - B_0 a + 2 = 0 \tag{4.2}$$

$$3C_0 a^2 + B_0 a(r-2) + (r-1)(r-2) = 0 \tag{4.3}$$

giving

$$(r+1)[(r-4) + B_0 a] = 0. \tag{4.4}$$

(B_0 and C_0 denote $B(x_0)$ and $C(x_0)$.)

If $C_0 = 0$, this means that $C = 0$ because x_0 is arbitrary in the SPA and

$$a = 2/B_0 \qquad r = -1, 2. \tag{4.5}$$

The (hybrid [4]) case $C \neq 0$ is a little more complicated. These are two solutions $a_\pm$ of (4.2) and, due to (4.4), two families of resonances -1, $r_\pm$ related by

$$a_\pm B_0 = 4 - r_\pm. \tag{4.6}$$

Inserting (4.6) into (4.2) we get

$$(8 - r_+ - r_-) = B_0^2/C_0 = \tfrac{1}{2}(4 - r_+)(4 - r_-) \tag{4.7}$$

or

$$r_- = 2 + 4/(r_+ - 2). \tag{4.8}$$

The resonance criterion requires that at least one of $r_\pm$ must be positive integer and the other must not be positive non-integer. The only such $r_\pm$ compatible with (4.8) are

$$r_\pm \in \{-2, 1, 3, 4, 6\}. \tag{4.9}$$

From (4.7) and (4.5) then we can conclude that the only Painlevé admissible dominant truncations from the class T $(-1, -3)$ are

$$T(-1, -3, R) = u_{xx} + (4 - R)b(x)uu_x + (2 - R)b(x)^2 u^3 \tag{4.10}$$

where $R = 1, 2, 3, 4$. The function $b(x)$ can be easily transformed out by $U(x) = b(x)u(x)$. These equations correspond to the cases i(b), i(c), i(d), i(e) of the Painlevé classification [1].

The resonance analysis of $T(-1, m)$ with lower m leads to Diophantine equations that are more complicated than (4.8). They are under investigation at present.

The dominant truncations of $T(-2, -4)$

$$Au_{xx} + Bu^2 \tag{4.11}$$

are not restricted by the resonance value criterion. It is easy to show that for arbitrary non-zero A and B there are two solutions -1 and 6 of (3.6).

On the other hand, we can exclude the whole class of dominant truncations $T(-3, -6)$. The equation for the resonances is

$$(r+1)(r^2 - 13r + 60) = 0 \tag{4.12}$$

that in addition to $r = -1$ has complex roots with Re $r > 0$ so that the resonance criterion excludes $T(-3, -6)$ from the list of candidates on dominant truncations of Painlevé-type equations. For reasons that will become clear in the next section we can exclude $T(-3, -7)$ as well.

The resonance analysis of $T(p, m)$ for $p < -1$ is slightly simpler than that of $T(-1, m)$. The dominant truncations of $T(p, m)$ are of the order $p - m$ in general and do not involve terms with the $(p-m-1)$th, $(p-m-2)$th, ..., $(2p-m+1)$th derivatives from which important identities for the resonances follow. One of them is [5]

$$r_1 + r_2 + \ldots + r_{p-m} = \tfrac{1}{2}(m-p)(p+m+1) \tag{4.13}$$

A convenient tool for the investigation of $T(p, m)$ with lower m are the so-called Painlevé chains investigated in the next section.

Let us stress that the SPA is useful not only for identifications of dominant parts $T(p, m, g)$ of Painlevé-type equations but also for the determination of their 'recessive' parts that contain terms with p dominance $> m$. We shall not deal with this problem in general here but we shall present an example.

The most general form of the ODE corresponding to $T(-2, -4)$ is

$$A(x)u_{xx} + B(x)u^2 + C(x)u_x + D(x)u + E(x) = 0. \tag{4.14}$$

By the transformation

$$z = \phi(x) \qquad W(z) = \lambda(x)u(x) + \mu(x) \tag{4.15}$$

this equation can be simplified to

$$W_{zz} = 6W^2 + S(z). \tag{4.16}$$

The compatibility condition at $r = 6$ gives

$$S(z) = az + b \tag{4.17}$$

which corresponds to cases II–IV of the Painlevé classification [1]. By similarly investigating the equations corresponding to (4.10) we obtain the other polynomial equations of this classification.

5. Painlevé chains

Painlevé chains were introduced in [4, 6] and it was observed that they have characteristic chains of resonances. The resonance formulae (3.6) and (3.1) enable us to explain these chains of resonances.

We shall define the derivative chain of the dominant truncation $T(p, m, g)$ as p-dominant truncations of the derivatives of $T(p, m, g)$, i.e. differentiating $T(p, m, g)$ we obtain

$$\frac{\mathrm{d}}{\mathrm{d}x} T(p, m, g) = \sum_{K \in M(p, m)} \frac{\mathrm{d}g_K}{\mathrm{d}x}[u]^K + g_K \frac{\mathrm{d}}{\mathrm{d}x}[u]^K \tag{5.1}$$

and obviously the p-dominant truncation ($p < 0$) of (5.1) is the sum of the second term. We shall denote $T_1(p, m, g) := d_x T(p, m, g)$ where d_x represents the differentiation of $[u]^K$ only (the $g_K(x)$ remain untouched by d_x).

Example. $d_x T(-2, -6, g) = T(-2, -7, \tilde{g})$, $d_x T(-3, -6, g) = T(-3, -7, \tilde{g})$ where $\tilde{g}$ is another analytic function (see table 1).

Definition. The nth element of the derivative chain is

$$T_n(p, m, g) := d_x^n T(p, m, g) := \sum_{K \in M(p,m)} g_K \frac{\mathrm{d}^n}{\mathrm{d}x^n}[u]^K. \tag{5.2}$$

We are interested in what happens to the set of resonances at this 'differentiation'. Let us start with $T_1(p, m, g)$. From (5.2) we obtain

$$T_1(p, m, g) = \sum_{K \in M(p,m)} g_K \sum_{j=0}^{N(K)} k_j [u]^{K_j} \tag{5.3}$$

where

$$K_j := (k_0, k_1, \ldots, k_j - 1, k_{j+1} + 1, \ldots, K_N). \tag{5.4}$$

One can immediately see from (2.7), (5.3) and (5.4) that

$$D(p, K_j) = D(p, K) - 1 \qquad j = 0, \ldots, N(K) \tag{5.5}$$

and therefore the p dominance of $T_1(p, m, g)$ is $m - 1$. For the other functions of K we obtain

$$d(K_j) = d(K) \tag{5.6}$$

$$[p]^{K_j} = (p - j)[p]^K \tag{5.7}$$

$$Y(p, K_j, r) = Y(p, K, r) + \frac{r}{p - j} \frac{[r + p]_j}{[p]_j}. \tag{5.8}$$

Due to (5.3), (5.6) and (5.7) the left-hand side of (3.1) transforms to

$$\begin{aligned} A_1(p, m, g, a) &= \sum_{K \in M(p,m)} g_K [p]^K a^{d(K)} \sum_{j=0}^{N(K)} (p - j) k_j \\ &= m A(p, m, g, a) \end{aligned} \tag{5.9}$$

so that the values of a that will enter the equation for resonances remain unchanged.

From (5.8) and definitions of $Y(p, K, r)$ and $M(p, m)$ we find that the left-hand side of the equation for resonances transforms to

$$\begin{aligned} R_1(p, m, g, a, r) &= \sum_{K \in M(p,m)} g_K [p]^K a^{d(K)} \sum_{j=0}^{N(K)} (p - j) k_j Y(p, K_j, r) \\ &= (r + m) R(p, m, g, a, r) \end{aligned} \tag{5.10}$$

Hence the set of resonance of $T_1(p, m, g)$ is that of $T(p, m, g)$ extended by $r = -m$ for every branch of a. By induction we get the following theorem.

Theorem 2. The p dominance of $T_n(p, m, g)$ is $m-n$ and the set of resonances for the solution a_j of (3.1) is

$$\mathcal{R}_n(a_j) = \mathcal{R}_0(a_j) \cup \{-m, -m+1, \ldots, -m+n-1\}. \tag{5.11}$$

The resonance values of the derivative chains investigated in [4-6] agree with equation (5.11).

A generalisation of the pure derivative chain is the chain where the operation of differentiation alternates with multiplication by $[u]^K$:

$$T_{K_1,K_2,\ldots,K_{s+1}}(p, m, g) := [u]^{K_{s+1}} d_x[u]^{K_s} d_x[u]^{K_{s-1}} \ldots d_x[u]^{K_1} T(p, m, g) \tag{5.12}$$

where K_j are arbitrary multi-indices (even the negative integer components are admissible). The special cases are the pure derivative chains where all $K_j = (0, 0, \ldots)$ and the Schwarzian chain investigated in [6]:

$$S_n(-1, -6) := u_1^{n+2} d_x^n [u_1^{-2}(2u_1u_3 - 3u_2^2)]. \tag{5.13}$$

For the investigation of the resonances of the generalised derivative chain we must investigate the behaviour of necessary characteristics at the multiplication of $T(p, m, g)$ by $[u]^K$. The equations for a and r remain unchanged and the p dominance changes to $m + D(p, K)$. From this and (5.10) it is easy to deduce the following theorem.

Theorem 3. The sets of resonances of (5.12) are

$$\begin{aligned}\mathcal{R}_{K_1,K_2,\ldots,K_{s+1}}&(a_i)\\ &= \mathcal{R}_0(a_i) \cup \{-m + D(p, K_1), -m+1+D(p, K_1)+D(p, K_2), \ldots,\\ &\quad -m+s-1+D(p, K_1)+\ldots+D(p, K_s)\}.\end{aligned} \tag{5.14}$$

As we have admitted also the negative powers of u_j at this stage we can see that, by the operation (5.12), one can produce rather arbitrary sets of resonances. The values of K_{s+1} do not appear in (5.14) so that the factor $[u]^{K_{s+1}}$ in (5.12) can be used for the repolynomialisation of the dominant truncation (5.12).

The importance of the derivative chains consists in the possibility of describing some dominant truncations with lower m as linear combinations of the generalised derivatives of those with higher m. An example of such an interesting generalised derivative chain is

$$[\rho_n(x)d_x + \lambda_n(x)] \ldots [\rho_1(x)d_x + \lambda_1(x)]u(x) \tag{5.15}$$

that for $n = 1, 2$ generates the general forms of $T(-1, -n-1)$ and for higher n their special cases. The resonance formulae (3.1) and (3.6) then enable a description of the sets of resonances similar to (5.11) and (5.14) and their exploitation for the identification of the Painlevé admissible dominant truncations.

6. Conclusions

We have introduced the concept of p dominance, useful for the determination of possible dominant truncations of Painlevé-type differential equations.

The resonance formulae, i.e. the general form of equations for the leading term and the resonances of the expansion (1.1), have proved useful for further classification of the dominant truncations and determination of resonances of the so-called Painlevé chains. The formulae may find other applications in methods based on singular point analysis.

In this paper we have restricted ourselves to ODE but the results are useful for the SPA of PDE [7] as well because quite often the dominant truncation of PDE contain derivatives WRT only one independent variable (KdV, MKdV and others). Moreover, it seems that the extension of the presented classification to PDE does not meet with major difficulties even though the calculations are much more tedious and extensive [8].

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